

A FEW EXPLICIT EXAMPLES OF COMPLEX DYNAMICS OF INERTIA GROUPS ON SURFACES - A QUESTION OF PROFESSOR IGOR DOLGACHEV

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Dedicated to Professor JongHae Keum on the occasion of his sixtieth birthday.

ABSTRACT. We give a few explicit examples which answer an open minded question of Professor Igor Dolgachev on complex dynamics of the inertia group of a smooth rational curve on a projective K3 surface and its variants for a rational surface and a non-projective K3 surface.

1. INTRODUCTION

In this note we work over \mathbb{C} . Our main results are Theorems 1.2, 1.3, 1.4, 1.5.

Let V be a smooth projective variety and C a subvariety of V . We define the subgroups of $\text{Aut}(V)$ called the *decomposition group* of C and the *inertia group* of C , respectively by

$$\text{Dec}(V) := \{f \in \text{Aut}(V) \mid f(C) = C\}, \quad \text{Ine}(V) := \{f \in \text{Dec}(V) \mid f|_C = \text{id}_C\}.$$

The aim of this note is to add some answers to the following open minded question asked by Professor Igor Dolgachev in his opening talk at the conference "Algebraic Geometry in honor of Professor JongHae Keum's 60-th birthday" and its variants:

Question 1.1. It is challenging to find (further) pairs (S, C) of a projective K3 surface and a smooth rational curve $C \subset S$ such that $\text{Ine}(C)$ contains an element with positive topological entropy.

Though it is not the definition, the *topological entropy* $h_{\text{top}}(f)$ of an automorphism $f \in \text{Aut}(V)$ is given by:

$$h_{\text{top}}(f) = \log \rho(f) \geq 0.$$

Here $\rho(f)$ is the spectral radius of the action of f on the group of the numerical equivalence classes of algebraic cocycles $\bigoplus_{p=0}^{\dim V} N^p(V)$ when V is projective. More generally, $\rho(f)$ is the spectral radius of the action of f on $\bigoplus_{p=0}^{\dim V} H^{p,p}(V)$, and these two quantities coincide if V is projective. Originally, $h_{\text{top}}(f)$ is defined as a fundamental measure "how fast forward orbits $\{f^n(x) \mid n \geq 1\}$, $\{f^n(y) \mid n \geq 1\}$ of two general points $x, y \in V$ spread out" and therefore it is considered as a fundamental measure of the complexity of complex dynamics given by the forward iterations of f (see [Gr87], [KH95], [DS05], [Tr15] for more details). We have $h_{\text{top}}(f) = 0$ if it is tame and $h_{\text{top}}(f) > 0$ if it is wild. So, Question 1.1 asks two extreme aspects; complex dynamics of f on $X \setminus C$ have to be rather wild, while it is

The author is supported by JSPS Grant-in-Aid (S) No 25220701, JSPS Grant-in-Aid (S) 15H05738, JSPS Grant-in-Aid (B) 15H03611, and by KIAS Scholar Program.

trivial on C . In a deeper level, the equidistribution of *isolated* fixed sets Q_n of f^n under $n \rightarrow \infty$ for any surface automorphism f of positive entropy is discovered in the survey [DS16, Theorem 5.6]. The difficulty in the proof is caused by fixed curves. Pairs (S, C) in Question 1.1 provide explicit examples of equidistribution theorem with fixed curves. This is also an interesting dynamical feature of Question 1.1. We remark that the higher dimensional case is an open problem (see eg. [DS16]).

Following [DZ01], we call a smooth rational surface W a *Coble surface* if $|-K_W| = \emptyset$ and $|-2K_W| \neq \emptyset$. Let $B \subset \mathbb{P}^2$ be a sextic plane curve with 10 nodes, T the blow up of \mathbb{P}^2 at the 10 nodes and $C \subset T$ the proper transform of B . Then $\mathbb{P}^1 \simeq C \in |-2K_T|$ and T is a Coble surface called *classical* ([DZ01]). T admits a finite double cover $\pi : S \rightarrow T$ branched along C . Then S is a projective K3 surface, which we call of *classical Coble type*. We denote the ramification curve $(\pi^*C)_{\text{red}} \subset S$ by the same letter C . In his talk, Professor Igor Dolgachev, among other things, showed that the pair (S, C) of a K3 surface S of classical Coble type and its ramification curve C gives an affirmative answer to Question 1.1. He shows first that (T, C) satisfies a similar property and then lifts to (S, C) .

The aim of this note is to remark the following theorems (Theorems 1.2, 1.3, 1.4, 1.5):

Theorem 1.2. *Question 1.1 is affirmative for a smooth Kummer surface $\text{Km}(E \times F)$ associated to the product abelian surface of elliptic curves E and F , i.e., there is a smooth rational curve $C \subset \text{Km}(E \times F)$ such that $\text{Ine}(C)$ has an element of positive topological entropy. If in addition E and F are not isogenous, then there is a smooth rational curve $C \subset \text{Km}(E \times F)$ satisfying the following two properties:*

- (1) $[\text{Aut}(\text{Km}(E \times F)) : \text{Dec}(C)] < \infty$ and $[\text{Dec}(C) : \text{Ine}(C)] = \infty$ and
- (2) There is $f \in \text{Ine}(C)$ with $h_{\text{top}}(f) > 0$.

The properties (1) and (2) claim that $\text{Dec}(C)$ is almost the same as $\text{Aut}(S)$ and $\text{Ine}(C)$ is much smaller than $\text{Dec}(C)$ but $\text{Ine}(C)$ yet enjoys rich dynamics as explained above. Note that if E and F are not isogenous, then $\text{Km}(E \times F)$ is a 2-elementary K3 surface in the sense of [Ni81], and at the same time, it is a finite double cover of a non-classical Coble surface in the sense of [DZ01]. So, it is very close to Professor Igor Dolgachev's example, even though our proof is completely different. We prove Theorem 1.2 in Section 4.

Let $\zeta_3 = (-1 + \sqrt{-3})/2 \in \mathbb{C}$ be a primitive third root of unity and E_3 be the elliptic curve of period ζ_3 . Then $E_3 \times E_3$ has an automorphism τ_3 of order three defined by

$$\tau_3^*(x, y) = (\zeta_3 x, \zeta_3^2 y) .$$

Here (x, y) are the standard coordinates of the universal covering space $\mathbb{C} \times \mathbb{C}$ of $E_3 \times E_3$. The minimal resolution S_3 of the quotient surface $(E_3 \times E_3)/\langle \tau_3 \rangle$ is a projective K3 surface called (one of) the most algebraic K3 surfaces ([Vi83]).

Theorem 1.3. *Question 1.1 is affirmative for the most algebraic K3 surface S_3 . More precisely, there is a smooth rational curve $C \subset S_3$ satisfying the following two properties:*

- (1) $[\text{Aut}(S_3) : \text{Dec}(C)] < \infty$ and $[\text{Dec}(C) : \text{Ine}(C)] = \infty$ and
- (2) There is $f \in \text{Ine}(C)$ with $h_{\text{top}}(f) > 0$.

As an application of Theorem 1.3, we also show the following:

Theorem 1.4. *There are a smooth rational surface W and a smooth rational curve $C \subset W$ such that*

- (1) $[\text{Aut}(W) : \text{Dec}(C)] < \infty$ and $[\text{Dec}(C) : \text{Ine}(C)] = \infty$;
- (2) There is $f \in \text{Ine}(C)$ with $h_{\text{top}}(f) > 0$; and
- (3) $|mK_W| = \emptyset$ for $m = -1$ and -2 , while $|-3K_W|$ consists of a unique element.

In particular, W is not isomorphic to any Coble surface.

We construct W as the minimal resolution of a suitable quotient of S_3 . Compare Theorem 1.4 with results due to McMullen [Mc07], rich rational surface dynamics arising from the decomposition group of the anti-canonical cuspidal curve and Bedford-Kim [BK09], rich rational surface dynamics with no stable curve. In Section 5, we prove Theorems 1.3 and derive Theorem 1.4 from Theorem 1.3.

Theorem 1.5. *Question 1.1 is affirmative for some K3 surface S with no non constant global meromorphic function, i.e., of algebraic dimension 0. More precisely, there are a K3 surface of algebraic dimension 0 and smooth rational curves $C \subset S$ and $D \subset S$ satisfying the following properties:*

- (1) $\text{Aut}(S) = \text{Dec}(C) = \text{Ine}(C)$, and there is $f \in \text{Ine}(C)$ with $h_{\text{top}}(f) > 0$;
- (2) $\text{Aut}(S) = \text{Dec}(D)$ and $\text{Ine}(D) = \{id_S\}$.

As a candidate S in Theorem 1.5, we choose a K3 surface constructed in [Og10]. Our proof is based on a very special feature occurring only on a non-projective K3 surface. We prove Theorem 1.5 in Section 6.

In Question 1.1, there are two essential issues to consider; "How to find an automorphisms of positive entropy" and "How to find a candidate curve C ". Our main approach in Theorems 1.2, 1.3 is to seek two different elliptic fibrations with positive Mordell-Weil rank. In Section 3, we briefly recall an existing method (Proposition 3.2) and synthesize it with the inertia group of some smooth rational curve in fibers (Proposition 3.4). Proposition 3.4 is the main tool in our proof of Theorems 1.2, 1.3. For Theorem 1.5, we use a non-symplectic automorphism of a K3 surface whose character of the action on the space of the global holomorphic 2-forms is not root of unity but a Galois conjugate of a Salem number, *a very special property occurring only on non-projective K3 surfaces*. It is worth recalling that this property also played an essential role in constructing automorphisms of (necessarily non-projective) K3 surfaces with Siegel disk by McMullen ([Mc02], see also [Og10]).

Acknowledgements. First of all, I would like to express my thanks to Professor Igor Dolgachev for his inspiring talk, interest in this work and valuable discussions. I would like to thank Professors Viacheslav Nikulin, Yuya Matsumoto, Shigeru Mukai, Matthias Schuett, Nessim Sibony, Xun Yu for valuable suggestions and comments and Professor JongHae Keum for his interest in this work. I would like to express my thanks to the organizers of Professor JongHae Keum's 60th birthday conference for invitation.

2. SOME BASIC TERMINOLOGIES FOR LATTICES AND K3 SURFACES

In this section, we briefly recall basic terminologies concerning lattices and K3 surfaces, which are used throughout this note.

We call a pair $(L, (*, **))$ of a free \mathbb{Z} -module L of positive finite rank r and a symmetric bilinear form $(*, **) : L \times L \rightarrow \mathbb{Z}$ a *lattice of rank r* . We often write the lattice $(L, (*, **))$ simply by L when no confusion arises. For any ring A , we denote by L_A the A -module

$L \otimes_{\mathbb{Z}} A$. We call the lattice $L = (L, (*, **))$ is hyperbolic (resp. negative definite) if the signature of the bilinear form $(*, **)$ is $(1, r - 1)$ and $r \geq 2$ (resp. $(0, r)$ and $r \geq 1$), considered as the real symmetric bilinear form on $L_{\mathbb{R}}$. We call a lattice L non-degenerate, if there is no $x \in L \setminus \{0\}$ such that $(x, y) = 0$ for all $y \in L$.

Let L be a hyperbolic lattice. Under the Euclidean topology of $L_{\mathbb{R}}$, the subset $\tilde{P}(L) := \{x \in L_{\mathbb{R}} \mid (x, x) > 0\}$ has two connected components. We choose and fix $P(L)$, which is any one of the two connected components of $\tilde{P}(L)$, and call it the positive cone. We denote by $O(L)$ the orthogonal group of a lattice L

$$O(L) := \{f \in \mathrm{GL}(L) \mid (f(x), f(y)) = (x, y) \ \forall x, y \in L\}$$

and by $O^+(L)$ the subgroup of $O(L)$ consisting $f \in O(L)$ such that $f_{\mathbb{R}} := f \otimes \mathrm{id}_{\mathbb{R}} \in \mathrm{GL}(L_{\mathbb{R}})$ preserves the positive cone $P(L)$. We denote the boundary of $P(L)$ in $L_{\mathbb{R}}$ by $\partial P(L)$. Then $f_{\mathbb{R}}(\partial P(L)) = \partial P(L)$ if $f \in O^+(L)$.

Let S be a K3 surface. We denote a nowhere vanishing global holomorphic 2-form on S by ω_S . By definition of K3 surface, ω_S is unique up to \mathbb{C}^\times . Important lattices in this note are the second cohomology lattice $H^2(S, \mathbb{Z})$, the Néron-Severi lattice $\mathrm{NS}(S)$ and the transcendental lattice $T(S) := \mathrm{NS}(S)^\perp$ in $H^2(S, \mathbb{Z})$ for a K3 surface S . The symmetric bilinear form on these lattices are the intersection form. $T(S)$ is also the minimal primitive sublattice of $H^2(S, \mathbb{Z})$ such that $\mathbb{C}\omega_S \subset T(S)_{\mathbb{C}}$. Note also that if $\mathrm{NS}(S)$ is non-degenerate, then $\mathrm{NS}(S) \oplus T(S) \subset H^2(S, \mathbb{Z})$ and $[H^2(S, \mathbb{Z}) : \mathrm{NS}(S) \oplus T(S)] < \infty$. We refer to [BHPV04] for basic facts on K3 surfaces and surfaces.

3. COMPLEX DYNAMICS ARISING FROM TWO DIFFERENT ELLIPTIC FIBRATIONS

Our main result of this section is Proposition 3.4.

Let X be a projective K3 surface. We call a surjective morphism $\varphi : X \rightarrow \mathbb{P}^1$ an elliptic fibration if the generic fiber X_η is an elliptic curve defined over the function field $\mathbb{C}(\mathbb{P}^1)$ of the base. We always choose the origin O of X_η in $X_\eta(\mathbb{C}(\mathbb{P}^1))$. The set $X_\eta(\mathbb{C}(\mathbb{P}^1))$ then forms an additive group $\mathrm{MW}(\varphi)$ with unit O , called the Mordell-Weil group of φ . As $\varphi : X \rightarrow \mathbb{P}^1$ is relatively minimal, the birational automorphism of X given by the translation on X_η by $P \in \mathrm{MW}(\varphi)$ is a biregular automorphism of X , i.e., $\mathrm{MW}(\varphi) \subset \mathrm{Aut}(X)$. By definition, $\mathrm{MW}(\varphi)$ is an abelian group and coincides with the group of translations of global sections of φ , under the natural bijective correspondence between $X_\eta(\mathbb{C}(\mathbb{P}^1))$ and the set of global sections of φ .

In this section, we are interested in a projective K3 surface in the following:

Set-up 3.1. X is a projective K3 surface admitting two different elliptic fibrations $\varphi_i : X \rightarrow \mathbb{P}^1$ ($i = 1, 2$) whose Mordell-Weil group $\mathrm{MW}(\varphi_i)$ has an element f_i of infinite order for each $i = 1, 2$.

Here two elliptic fibrations φ_i ($i = 1, 2$) are said to be different if the generic point of the generic fiber, in the sense of scheme, are different (non-closed) points of X .

We say that a property (P) (concerning closed points of X) holds for very general points if there is a countable union B of proper closed subvarieties of X such that the property (P) holds for any closed point $x \in X \setminus B$.

K3 surfaces in Set-up 3.1 have rich complex dynamical properties, as the next proposition shows:

Proposition 3.2. *Under Set-up 3.1, the following holds:*

- (1) *The orbit $\langle f_1, f_2 \rangle \cdot x$ is dense in X for very general points $x \in X$. Here topology of X is not the Zariski topology but the Euclidean topology (cf. [Mc02, Questions, Page 206] for a relevant open question).*
- (2) *There is a positive integer n such that $\langle f_1^n, f_2^n \rangle = \langle f_1^n \rangle * \langle f_2^n \rangle \simeq \mathbb{Z} * \mathbb{Z}$ (the free product).*
- (3) *There is $f \in \langle f_1, f_2 \rangle$ such that $h_{\text{top}}(f) > 0$.*

Proof. The assertion (1) is observed by [Ca01]. The assertion (2) is proved by [Og08] in a more general setting and is also related to Tits' alternative for the automorphism group of a compact Kähler manifold ([Zh09] and references therein).

The assertion (3) is implicit in [Og07]. As the assertion (3) is crucial in this note, we shall give a complete proof here.

Let $v_i \in \text{NS}(X)$ be the fiber class of φ_i ($i = 1, 2$). Then v_i ($i = 1, 2$) are primitive integral nef classes and they are \mathbb{Q} -linearly independent in $\text{NS}(X)_{\mathbb{Q}}$ as φ_i are different.

Now assuming to the contrary that $h_{\text{top}}(f) = 0$ for all $f \in \langle f_1, f_2 \rangle$, we shall derive a contradiction. We use the following theorem proved in [Og07]:

Theorem 3.3. *Let L be a hyperbolic lattice and $G < \text{O}^+(L)$ a subgroup of $\text{O}^+(L)$. Assume that the natural logarithm of spectral radius of g is 0 for all $g \in G$. Then, there are a subgroup N of G such that $[G : N] < \infty$ and a primitive integral element $v \in L \cap \partial P(L) \setminus \{0\}$ such that $g(v) = v$ for all $g \in N$.*

As X is a smooth projective surface, $\text{NS}(X)$ is a hyperbolic lattice of signature $(1, \rho(X) - 1)$. Here $\rho(X)$ is the Picard number of X and $\rho(X) \geq 2$ as an ample class of X and the fiber class v_1 are \mathbb{Q} -linearly independent in $\text{NS}(X)_{\mathbb{Q}}$. As usual, we choose the positive cone $P(\text{NS}(X))$ so that $P(\text{NS}(X))$ contains the ample cone of X . Then $\langle f_1, f_2 \rangle$ preserves the positive cone, i.e., $\langle f_1^*, f_2^* \rangle < \text{O}^+(\text{NS}(X))$. Here f^* is the natural action of f on $\text{NS}(X)$. As we assume that $h_{\text{top}}(f) = 0$ for all $f \in \langle f_1, f_2 \rangle$, i.e., the natural logarithm of the spectral radius of $f_{\mathbb{C}}^*$ is 0 for all $f^* \in \langle f_1^*, f_2^* \rangle$, it follows from Theorem 3.3 that there is a finite index subgroup N of $\langle f_1^*, f_2^* \rangle$ and an integral primitive element $v \in \text{NS}(X) \cap \partial P(\text{NS}(X)) \setminus \{0\}$ such that $f^*(v) = v$ for all $f \in N$. As N is a finite index subgroup of $\langle f_1^*, f_2^* \rangle$ and $f_1^*, f_2^* \in \langle f_1^*, f_2^* \rangle$, there is a positive integer n such that $(f_1^*)^n, (f_2^*)^n \in N$. Then $(f_i^*)^n(v_i) = v_i$ and $(f_i^*)^n(v) = v$. As v_1 and v_2 are \mathbb{Q} -linearly independent in $\text{NS}(X)_{\mathbb{Q}}$, it follows that either v_1 and v are \mathbb{Q} -linearly independent or v_2 and v are \mathbb{Q} -linearly independent. By changing the order if necessary, we may assume that v_1 and v are \mathbb{Q} -linearly independent. Then $(v_1 + v)^2 > 0$ by the Hodge index theorem. As $(f_i^*)^n(v_1 + v) = v_1 + v$, it follows that $(f_i^*)^n$ is of finite order on $\text{NS}(X)$. This is because the orthogonal complement of $v_1 + v$ in $\text{NS}(X)$ is negative definite by the Hodge index theorem and it is preserved by $(f_i^*)^n$. (Note that $\text{O}(L, (*, **))$ is finite if the integral bilinear form $(*, **)$ is negative definite.)

As X is a *projective* K3 surface, it follows that f_1 is of finite order also as an element of $\text{Aut}(X)$. This is because f_1 is then an automorphism of X as a polarized manifold and X has no global vector field. However, this contradicts to our assumption that f_1 is of infinite order in $\text{MW}(\varphi_1) < \text{Aut}(X)$. \square

The next proposition synthesizes Proposition 3.2 (3) and the inertia group of a smooth rational curve that we are looking for:

Proposition 3.4. *Under Set-up 3.1, we assume further that there is a smooth rational curve $C \subset X$ such that φ_i has a singular fiber F_i in which C is an irreducible component meeting at least three other irreducible components of F_i , i.e., F_i is of Kodaira type I_b^* , II^* , III^* or IV^* for both $i = 1$ and 2 . Then there is $f \in \text{Ine}(C)$ such that $h_{\text{top}}(f) > 0$. (See [Ko63, Page 565] for the notation. For instance, if F_1 is of Kodaira type IV^* and F_2 is of Kodaira type III^* , then C is the irreducible component of multiplicity 3 of F_1 and at the same time the irreducible component of multiplicity 4 of F_2 .)*

Proof. Let C_{ij} ($1 \leq j \leq n_i$) be the set of irreducible components of F_i ($i = 1, 2$). Then we have a natural group homomorphism

$$\tau_i : \text{MW}(\varphi_i) \rightarrow \text{Aut}_{\text{set}}(\{S_{ij}\}_{1 \leq j \leq n_i}) \simeq \Sigma_{n_i}.$$

Here Σ_n is the symmetric group of n letters. As n_i is a natural number, $\text{Ker } \tau_i$ is a finite index subgroup of $\text{MW}(\varphi_i)$. Recall that $\text{MW}(\varphi_i)$ ($i = 1, 2$) has an element of infinite order. Then there is $f_i \in \text{Ker } \tau_i$ such that f_i is of infinite order as well. By definition, we have $f_i(C_{ij}) = C_{ij}$ for all j . Thus $f_i(C) = C$ and $f_i(P_{i,k}) = P_{i,k}$ ($k = 1, 2, 3$). Here $P_{i,k} \in C$ ($k = 1, 2, 3$) are the intersection points of C with other (at least) three irreducible components of F_i . As $C \simeq \mathbb{P}^1$ and $P_{i,k}$ ($k = 1, 2, 3$) are mutually distinct closed points on C , it follows that $f_i|_C = \text{id}_C$, i.e., $f_i \in \text{Ine}(C)$. Hence

$$\langle f_1, f_2 \rangle \subset \text{Ine}(C).$$

By Proposition 3.2 (3), there is $f \in \langle f_1, f_2 \rangle$ such that $h_{\text{top}}(f) > 0$. This f satisfies all the requirements. \square

4. PROOF OF THEOREM 1.2

In this section, we prove Theorem 1.2.

Throughout this section, we denote by $S = \text{Km}(E \times F)$ the Kummer surface associated to the product of two elliptic curves E and F .

Let $\{P_i\}_{i=1}^4$ (resp. $\{Q_i\}_{i=1}^4$) be the 2-torsion subgroup of F (resp. E). Then S contains 24 *visible* smooth rational curves. They are 8 smooth rational curves E_i, F_i ($1 \leq i \leq 4$) arising from 8 elliptic curves $E \times \{P_i\}, \{Q_i\} \times F$ on $E \times F$ and 16 exceptional curves C_{ij} over the 16 singular points of type A_1 on the quotient surface $E \times F / \langle -\text{id}_{E \times F} \rangle$. We use the same notation as in [Og89, Page 655]. See Figure 1 for the configuration of these 24 visible smooth rational curves on S .

Throughout this section, we set:

$$C := E_4 \subset S.$$

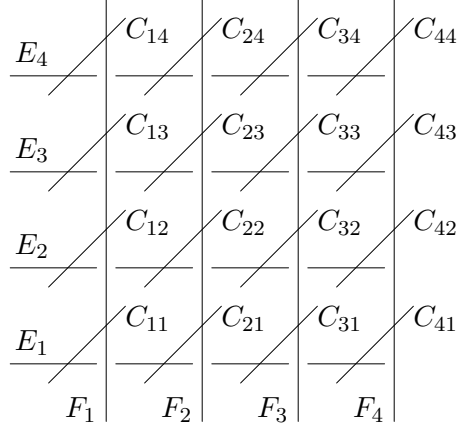
Proposition 4.1. $[\text{Dec}(C) : \text{Ine}(C)] = \infty$ and there is $f \in \text{Ine}(C)$ such that $h_{\text{top}}(f) > 0$.

Proof. Consider the following three divisors of Kodaira type IV^* on S :

$$\begin{aligned} D_1 &:= F_1 + 2C_{14} + F_2 + 2C_{24} + F_3 + 2C_{34} + 3E_4, \\ D_2 &:= F_1 + 2C_{14} + F_2 + 2C_{24} + F_4 + 2C_{44} + 3E_4, \\ D_3 &:= E_4 + 2C_{44} + E_3 + 2C_{43} + E_2 + 2C_{42} + 3F_4. \end{aligned}$$

Then the linear system $|D_k|$ is a free pencil and defines an elliptic fibration

$$\varphi_k : S \rightarrow \mathbb{P}^1$$

FIGURE 1. Curves E_i , F_j and C_{ij}

with a singular fiber D_k of Kodaira type IV^* , for each $k = 1, 2, 3$. Notice that the three fibrations φ_k are mutually different and each φ_k admits a section as we shall see.

First, we show that $[\text{Dec}(E_4) : \text{Ine}(E_4)] = \infty$. Recall that we set $C = E_4$.

The curves C_{14} and C_{24} are disjoint section of φ_3 meeting the same irreducible component E_4 of the fiber D_3 of φ_3 . We choose C_{14} as the zero section of φ_3 . Then the section C_{24} defines $f_3 \in \text{MW}(\varphi_1)$, which preserves each irreducible component of D_3 . In particular, $f_3(E_4) = E_4$. Note that

$$E_4^0 := E_4 \setminus C_{44} = \mathbb{C}.$$

Then, by [Ko63, Table 1, Page 604], if we set

$$0 := C_{14} \cap E_4^0 \subset E_4^0 = \mathbb{C},$$

$$a := C_{24} \cap E_4^0 \subset E_4^0 = \mathbb{C},$$

then $f_3(E_4) = E_4$ and f_3 acts on $E_4^0 = \mathbb{C}$ by the addition $x \mapsto x + a$. As $C_{14} \cap C_{24} = \emptyset$, it follows that $a \neq 0$, hence f_3 is of infinite order on E_4 . As $f_3 \in \text{Dec}(E_4)$ and $f_3|_{E_4}$ is of infinite order on E_4 , it follows that the class f_3 is of infinite order in the quotient group $\text{Dec}(E_4)/\text{Ine}(E_4)$. Hence $[\text{Dec}(E_4) : \text{Ine}(E_4)] = \infty$.

Next we show that there is $f \in \text{Ine}(C)$ such that $h_{\text{top}}(f) > 0$.

Note that $C = E_4$ is the irreducible component of both D_1 and D_2 meeting three other irreducible components of both D_1 and D_2 . Note also that C_{11} and C_{12} are disjoint section of both φ_k ($k = 1, 2$), meeting the same irreducible component F_1 of the fiber D_k . Thus, for the same reason above, C_{11} and C_{12} define the element $f_k \in \text{MW}(\varphi_k)$ of infinite order for each $k = 1$ and 2 . As $C = E_4$ is the irreducible component of both D_k ($k = 1, 2$) meeting three other irreducible components of D_k , there is then $f \in \text{Ine}(E_4)$ such that $h_{\text{top}}(f) > 0$ by Proposition 3.4. \square

The following proposition completes the proof of Theorem 1.2:

Proposition 4.2. *Let $S = \text{Km}(E \times F)$ and $\mathbb{P}^1 \simeq C = E_4 \subset S$ as before. Assume that the elliptic curves E and F are not isogenous. Then $[\text{Aut}(S) : \text{Dec}(C)] < \infty$.*

Proof. Consider the element $\iota \in \text{Aut}(S)$ of order 2, induced by the element $(-id_E, id_F) \in \text{Aut}(E \times F)$. As E and F are not isogenous, it follows from [Og89, Lemma 1.4] that ι is in the center of $\text{Aut}(S)$ and the pointwisely fixed locus of ι is

$$S^\iota = \cup_{i=1}^4 E_i \cup \cup_{i=1}^4 F_i .$$

Thus S^ι is preserved by $\text{Aut}(S)$ and we obtain a natural group homomorphism

$$\tau : \text{Aut}(S) \rightarrow \text{Aut}_{\text{set}}(\{E_i, F_i\}_{i=1}^4) \simeq \Sigma_8 .$$

Here, as before Σ_n is the symmetric group of n letters. As Σ_8 is a finite group, it follows that $[\text{Aut}(S) : \text{Ker } \tau] < \infty$. By definition of $\text{Dec}(E_4)$, we have

$$\text{Ker } \tau < \text{Dec}(E_4) < \text{Aut}(S) .$$

Thus $[\text{Aut}(S) : \text{Dec}(E_4)] < \infty$ as well. \square

Remark 4.3. Let $S := \text{Km}(E \times E)$. Then $[\text{Aut}(S) : \text{Dec}(C)] = \infty$ ($C = E_4$). This is observed as follows. Let $\Delta \in \text{Aut}(E \times E)$ defined by $(x, y) \mapsto (x, y + x)$. Consider the automorphism $\delta \in \text{Aut}(S)$ induced by Δ . Then δ is of infinite order but $\delta^n(E_4) = E_4$ only if $n = 0$.

Remark 4.4. A projective K3 surface V is called *2-elementary* if V has an automorphism ι of order 2 such that $\iota^* \omega_V = -\omega_V$, or equivalently $\iota^*|_{T(S)} = -id_{T(V)}$, and $\iota^* = id$ on $\text{NS}(V)$. The name comes from the fact that the discriminant group $\text{NS}(V)^*/\text{NS}(V)$ is then isomorphic to some 2-elementary group $(\mathbb{Z}/2)^{\oplus a(V)}$. 2-elementary K3 surfaces are intensively studied by Nikulin ([Ni81]9. The involution ι is in the center of $\text{Aut}(V)$ and the fixed locus of ι is preserved under $\text{Aut}(V)$. By the classification in [Ni81], if a 2-elementary K3 surface has a smooth rational curve C and an automorphism of positive entropy, then V^ι has to be either empty or consists of smooth rational curves (which are necessarily disjoint). The second case occurs exactly when $\rho(V) + a(V) = 22$ by [Ni81]. Kummer surfaces $\text{Km}(E \times F)$ of non-isogenous E and F and K3 surfaces of classical Coble type are two special cases of 2-elementary K3 surfaces with $(\rho(V), a(V)) = (18, 4)$, $(11, 11)$ respectively. It may be interesting to check if any 2-elementary K3 surfaces with $\rho(V) + a(V) = 22$ has a smooth rational curve C whose inertia group has an automorphism of positive entropy or not. See also Remark 5.5.

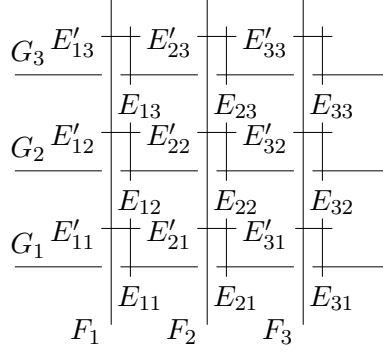
5. PROOF OF THEOREMS 1.3 AND 1.4

In this section, we show Theorems 1.3 and 1.4.

Throughout this section, $\zeta_3 := (-1 + \sqrt{-3})/2$ is a primitive 3rd root of unity, $E = E_3$ is the elliptic curve of period ζ_3 and $S = S_3$ is the minimal resolution of the quotient surface $(E \times E)/\langle (\zeta_3, \zeta_3^2) \rangle$. Then S is a projective K3 surface with 24 *visible* smooth rational curves F_i, G_i ($i = 1, 2, 3$), E_{ij}, E'_{ij} ($i = 1, 2, 3, j = 1, 2, 3$) whose configuration is in Figure 2. We follow [OZ96, Page 1281] for the notation of these 24 curves.

Throughout this section, we also denote by $g = g_3$ the automorphism of S of order 3, induced by the automorphism $\tilde{g} \in \text{Aut}(E \times E)$ defined by

$$\tilde{g}^*(x, y) = (x, \zeta_3 y) .$$

FIGURE 2. Curves F_i , G_j , E_{ij} and E'_{ij}

Here (x, y) are the standard coordinates of the universal covering space $\mathbb{C} \times \mathbb{C}$ of $E \times E$. Then by an explicit computation or by using [OZ96, Example 1, Theorem 3, Lemma 2.3], we have the following:

Proposition 5.1. *The pair (S, g) satisfies the following:*

- (1) *Each of the 24 visible smooth rational curves above, say D , is preserved by g , i.e., $g \in \text{Dec}(D)$. Moreover the fixed locus X^g consists of 6 smooth rational curves F_i , G_i and 9 isolated points $P_{ij} = E_{ij} \cap E'_{ij}$ ($1 \leq i, j \leq 3$).*
- (2) *$\rho(S) = 20$, $g^* = \text{id}$ on $\text{NS}(S)$ and $g^*\omega_S = \zeta_3\omega_S$.*

Set $\text{Aut}^0(S) := \{f \in \text{Aut}(S) \mid f^*\omega_S = \omega_S\}$. As S is projective, the pluricanonical representation of $\text{Aut}(S)$ is of finite order ([Ue75, Theorem 14.10]). Thus $\text{Aut}^0(S)$ is a normal subgroup of $\text{Aut}(S)$ of finite index.

From now until the end of this section, we set

$$C = G_3 \simeq \mathbb{P}^1.$$

We show that C satisfies the requirements in Theorem 1.3.

Proposition 5.2. $[\text{Aut}(S) : \text{Dec}(C)] < \infty$.

Proof. As $\text{Aut}^0(S)$ is a normal subgroup of $\text{Aut}(S)$ of finite index, it suffices to show that

$$[\text{Aut}^0(S) : \text{Aut}^0(S) \cap \text{Dec}(C)] < \infty.$$

Indeed, for a group G and a normal subgroup $N < G$ and a subgroup $H < G$, we have an obvious inequality:

$$[G : W] \leq [N : N \cap W] \cdot [G : N].$$

As $g^* = \text{id}$ on $\text{NS}(S)$ by Proposition 5.1 (2), and $f^* = \text{id}$ on $T(S)$ for $f \in \text{Aut}^0(S)$, it follows that $(f \circ g)^* = (g \circ f)^*$ on $\text{NS}(S) \oplus T(S)$ for $f \in \text{Aut}^0(S)$. As $\text{NS}(S) \oplus T(S)$ is a finite index subgroup of $H^2(S, \mathbb{Z})$, it follows that $(f \circ g)^* = (g \circ f)^*$ on $H^2(S, \mathbb{Z})$.

Hence $f \circ g = g \circ f$ by the global Torelli theorem for K3 surfaces. Then $f(S^g) = S^g$ for $f \in \text{Aut}^0(S)$. Therefore, by Proposition 5.1 (1), we have a natural group homomorphism

$$\tau : \text{Aut}^0(S) \rightarrow \text{Aut}_{\text{set}}(\{F_i, G_i\}_{i=1}^3) \simeq \Sigma_6 .$$

As $\text{Ker } \tau$ is of finite index in $\text{Aut}^0(S)$ and

$$\text{Ker } \tau < \text{Dec}(G_3) \cap \text{Aut}^0(S) < \text{Aut}^0(S) ,$$

the result follows. \square

Proposition 5.3. $[\text{Dec}(C) : \text{Ine}(C)] = \infty$. *More strongly, there is $h \in \text{Dec}(C)$ such that $h|_C$ is of infinite order and $h \in \text{Aut}^0(S)$.*

Proof. Consider the elliptic fibration $\varphi : S \rightarrow \mathbb{P}^1$ given by the divisor D of Kodaira type III^* :

$$D := G_3 + 2E_{33} + 3E'_{33} + 4F_3 + 3E'_{31} + 2E_{31} + G_1 + 2E'_{32} .$$

Then E_{13} and E_{23} are disjoint section of φ meeting the singular fiber D of φ of Kodaira type III^* at the same irreducible component G_3 . We choose E_{13} as the zero section of φ . Let $h \in \text{MW}(\varphi)$ be the element defined by E_{23} . Then $h \in \text{Dec}(C)$ and $h|_C$ is of infinite order for the same reason as in the proof of Proposition 4.1. Hence $[\text{Dec}(C) : \text{Ine}(C)] = \infty$, again for the same reason as in the proof of Proposition 4.1. We have also $h^*\omega_S = \omega_S$ as $h \in \text{MW}(\varphi)$. \square

Proposition 5.4. *There is $f \in \text{Ine}(C)$ such that $h_{\text{top}}(f) > 0$ and $f \in \text{Aut}^0(S)$.*

Proof. Consider the elliptic fibrations $\varphi_k : S \rightarrow \mathbb{P}^1$ ($k = 1, 2$) given respectively by the divisors of Kodaira type III^*

$$D_1 := F_1 + 2E'_{13} + 3E_{13} + 4G_3 + 3E_{23} + 2E'_{23} + F_2 + 2E_{33} ,$$

$$D_2 := F_1 + 2E'_{13} + 3E_{13} + 4G_3 + 3E_{33} + 2E'_{33} + F_3 + 2E_{23} .$$

Then E'_{11} and E'_{12} are disjoint sections of both φ_k meeting the irreducible component F_1 of the singular fiber D_k of φ_k . We choose E'_{11} as the zero section of both φ_k and consider the element $f_k \in \text{MW}(\varphi_k)$ defined by the section E'_{12} . Then for the same reason as in the proof of Proposition 4.1, f_k is of infinite order in $\text{MW}(\varphi_k)$ and is also an element of $\text{Ine}(G_3)$. Note that $C = G_3$ is the irreducible component of D_k meeting three other irreducible component of D_k for both $k = 1$ and 2 . Then there is $f \in \langle f_1, f_2 \rangle$ with $h_{\text{top}}(f) > 0$ by Proposition 3.4. Note also that $f^*\omega_S = \omega_S$. This is because $f_k^*\omega_S = \omega_S$ as $f_k \in \text{MW}(\varphi_k)$ ($k = 1, 2$). This completes the proof. \square

Now Theorem 1.3 has been proved.

Remark 5.5. As $\text{NS}(S_3)^*/\text{NS}(S_3) \simeq \mathbb{Z}/3$ ([Vi83]), the most algebraic K3 surface S_3 is not a 2-elementary K3 surface in the sense of [Ni81]. As it is pointed by Xun Yu, by using a similar method, one can show that the other most algebraic K3 surface S_4 ([Vi83]) also satisfies the same property as in Theorem 1.3. We leave details to the readers. We note that S_4 is a 2-elementary K3 surface with $(\rho(S_4), a(S_4)) = (20, 2)$ (see Remark 4.4 for $a(S_4)$) under a non-symplectic involution g_2 and the quotient surface $S_4/\langle g_2 \rangle$ is a smooth non-classical Coble surface.

Next we show Theorem 1.4. We continue to use the same notations introduced at the beginning of this section.

Let $\nu : W \rightarrow \overline{W}$ be the minimal resolution of the quotient surface $\overline{W} := S/\langle g \rangle$ and $\pi : S \rightarrow \overline{W}$ the quotient map. By the description of S^g , the surface \overline{W} has exactly 9 singular points of type $\frac{1}{3}(1, 1)$ at $\pi(P_{jk})$. Then $C_{jk} := \nu^{-1}(\pi(P_{jk}))_{\text{red}}$ is a smooth rational curve with $(C_{jk}, C_{jk}) = -3$. We denote the smooth locus of \overline{W} by \overline{W}^0 , i.e., $\overline{W}^0 := \overline{W} \setminus \{\pi(P_{jk})\}_{j,k=1}^3$. We set $\overline{F}_i := \pi(F_i)_{\text{red}}$ and $\overline{G}_i := \pi(G_i)_{\text{red}}$ ($1 \leq i \leq 3$). These 6 curves are mutually disjoint smooth rational curves in \overline{W}^0 with self-intersection number -6 . As $\overline{F}_i, \overline{G}_i \subset \overline{W}^0$, the morphism ν is isomorphic around \overline{F}_i and \overline{G}_i . For this reason, we denote $\nu^*(\overline{F}_i)$ and $\nu^*(\overline{G}_i)$ by the same letter \overline{F}_i and \overline{G}_i .

Proposition 5.6. *The surface W is a smooth rational surface such that $|mK_W| = \emptyset$ for $m = -1$ and -2 and $|-3K_W|$ consists of a unique element. More precisely,*

$$|-3K_W| = \left\{ 2 \sum_{i=1}^3 (\overline{F}_i + \overline{G}_i) + \sum_{j,k=1}^3 C_{jk} \right\}.$$

Proof. As $q(S) = 0$ and $\nu^{-1} \circ \pi : S \dashrightarrow W$ is a dominant rational map between smooth varieties, it follows that $q(W) = 0$. Here $q(W)$ and $q(S)$ are the irregularities of W and S .

As π is a finite cyclic cover of degree 3, being totally ramified along $\sum_{i=1}^3 (F_i + G_i)$ over \overline{W}^0 and K_S is trivial, it follows from the ramification formula that $-K_{\overline{W}}$ is \mathbb{Q} -linearly equivalent to an effective \mathbb{Q} -divisor:

$$\frac{2}{3} \sum_{i=1}^3 (\overline{F}_i + \overline{G}_i) \sim_{\mathbb{Q}} -K_{\overline{W}}.$$

In particular, $mK_{\overline{W}} \not\sim_{\mathbb{Q}} 0$ for any integer $m \neq 0$. Here and hereafter, we denote by $\sim_{\mathbb{Q}}$ the \mathbb{Q} -linear equivalence of Weil divisors.

Assume that $|mK_W| \neq \emptyset$ for some integer $m \neq 0$.

Then there is $D \in |mK_W|$. Let $\overline{D} := \nu_* D$. Then \overline{D} is a *non-zero* effective Weil divisor on \overline{W} and it is linearly equivalent to $mK_{\overline{W}}$, as \overline{W} is normal and ν is isomorphic over \overline{W}^0 . We also note that as \overline{W} has only quotient singularities, any Weil divisor on \overline{W} is \mathbb{Q} -Cartier.

As \overline{W} is normal and projective and $-K_{\overline{W}}$ is \mathbb{Q} -linearly equivalent to a non-zero effective divisor, it follows that $m < 0$. In particular, $|mK_W| = \emptyset$ for all positive integer $m > 0$. As in addition $q(W) = 0$, W is a rational surface by Castelnuovo's criterion.

From now, we assume that $m < 0$. Set $n = -m$. Then n is a positive integer. Assume that $M \in |-nK_{\overline{W}}|$. Note that $-nK_{\overline{W}}$ is \mathbb{Q} -linearly equivalent to an effective \mathbb{Q} -divisor

$$\frac{2n}{3} \sum_{i=1}^3 (\overline{F}_i + \overline{G}_i).$$

As the 6 curves \overline{F}_i and \overline{G}_i are mutually disjoint curves with $(\overline{F}_i, \overline{F}_i) = (\overline{G}_i, \overline{G}_i) = -6 < 0$, it follows that $(M, \overline{F}_i) < 0$ and $(M, \overline{G}_i) < 0$ for $i = 1, 2, 3$. Then

$$\overline{F}_i, \overline{G}_i \subset \text{Supp } M,$$

as M is an effective Weil divisor. Hence

$$M_1 := M - \sum_{i=1}^3 (\overline{F}_i + \overline{G}_i) \left(\sim_{\mathbb{Q}} \frac{2n-3}{3} \sum_{i=1}^3 (\overline{F}_i + \overline{G}_i) \right)$$

is an effective Weil divisor. Thus $n \geq 2$. If $n \geq 2$, then $(M_1, \overline{F}_i) < 0$ and $(M_1, \overline{G}_i) < 0$. Therefore, for the same reason above,

$$M_2 := M_1 - \sum_{i=1}^3 (\overline{F}_i + \overline{G}_i) = M - 2 \sum_{i=1}^3 (\overline{F}_i + \overline{G}_i) \left(\sim_{\mathbb{Q}} \frac{2n-6}{3} \sum_{i=1}^3 (\overline{F}_i + \overline{G}_i) \right)$$

is also an effective divisor. Thus $n \geq 3$ and if $n = 3$, then $M_2 = 0$ and therefore $|-3K_{\overline{W}}|$ consists of the single element $2 \sum_{i=1}^3 (\overline{F}_i + \overline{G}_i)$.

Hence, by the observation made at the beginning, $|-K_W| = \emptyset$ and $|-2K_W| = \emptyset$, and moreover if $D \in |-3K_W|$ then

$$\nu_* D = 2 \sum_{i=1}^3 (\overline{F}_i + \overline{G}_i)$$

as a divisor. Hence we have

$$D = 2 \sum_{i=1}^3 (\overline{F}_i + \overline{G}_i) + \sum_{j,k=1}^3 a_{jk} C_{jk}$$

for some non-negative integers a_{jk} as a divisor. Here, the integers a_{jk} are uniquely determined, as C_{jk} are mutually disjoint curve with $(C_{jk}, C_{jk}) = -3$ and $(C_{jk}, \overline{F}_i) = (C_{jk}, \overline{G}_i) = 0$. Hence the element $|-3K_W|$ is unique if exists. On the other hand, as ν is a minimal resolution of 9 singular points of type $1/3(1,1)$, we know by the adjunction formula that $-3K_W$ is linearly equivalent to an effective divisor

$$2 \sum_{i=1}^3 (\overline{F}_i + \overline{G}_i) + \sum_{j,k=1}^3 C_{jk}.$$

This completes the proof. \square

Let $\overline{C} := \overline{G}_3 \subset W$. Then \overline{C} is a smooth rational curve on W . The next proposition completes the proof of Theorem 1.4.

Proposition 5.7. *The curve \overline{C} satisfies*

- (1) $[\text{Aut}(W) : \text{Dec}(\overline{C})] < \infty$ and $[\text{Dec}(\overline{C}) : \text{Ine}(\overline{C})] = \infty$.
- (2) *There is $f_W \in \text{Ine}(\overline{C})$ such that $h_{\text{top}}(f_W) > 0$.*

Proof. As $\text{Aut}(W)$ acts on $|-3K_W|$, the divisor

$$2 \sum_{i=1}^3 (\overline{F}_i + \overline{G}_i) + \sum_{j,k=1}^3 C_{jk}$$

is stable under $\text{Aut}(W)$ by Proposition 5.6. Therefore we have a natural representaion

$$\tau : \text{Aut}(W) \rightarrow \text{Aut}_{\text{set}}(\{\overline{F}_i, \overline{G}_i\}_{i=1}^3) \simeq \Sigma_6.$$

Then $[\text{Aut}(W) : \text{Ker } \tau] < \infty$. As $\text{Ker } \tau < \text{Dec}(\overline{G}_3) < \text{Aut}(W)$, we have $[\text{Aut}(W) : \text{Dec}(\overline{G}_3)] < \infty$.

We show that $[\text{Dec}(\overline{C}) : \text{Ine}(\overline{C})] = \infty$. Consider the element $h \in \text{Aut}(S)$ in Proposition 5.3. Recall that $g^* = \text{id}$ on $\text{NS}(S)$ and $h^* = \text{id}$ on $T(S)$ as $h \in \text{Aut}^0(S)$. Thus $h \circ g = g \circ h$ for the same reason as in the proof of Proposition 5.2. It follows that h induces an automorphism of $h_W \in \text{Aut}(W)$. By definition of h_W , the map $\nu^{-1} \circ \pi : S \dashrightarrow W$ is equivariant under the actions h and h_W . As $\overline{C} = \nu^{-1} \circ \pi(G_3)$, $h(G_3) = G_3$ and $h|_{G_3}$ is of infinite order in $\text{Aut}(G_3)$, it follows that $h_W(\overline{C}) = \overline{C}$ and $h_W|_{\overline{C}}$ is of infinite order in $\text{Aut}(\overline{C})$. Hence the class of h_W is of infinite order in the quotient group $\text{Dec}(\overline{C})/\text{Ine}(\overline{C})$. This shows $[\text{Dec}(\overline{C}) : \text{Ine}(\overline{C})] = \infty$.

We show the assertion (2). Let $f \in \text{Aut}(S)$ be an automorphism found in Proposition 5.4. As $g^* = \text{id}$ on $\text{NS}(S)$ and $f \in \text{Aut}^0(S)$, for the same reason as above, f induces an automorphism $f_W \in \text{Aut}(W)$ of W . As $f \in \text{Ine}(G_3)$, i.e., $f|_{G_3} = \text{id}_{G_3}$, we have $f_W|_{\overline{C}} = \text{id}_{\overline{C}}$, i.e., $f_W \in \text{Ine}(\overline{C})$, again for the same reason as above. Moreover, as $\nu^{-1} \circ \pi : S \dashrightarrow W$ is a generically finite dominant rational map which is equivariant under the actions of f and $f|_W$, we have

$$h_{\text{top}}(f_W) = h_{\text{top}}(f) > 0$$

by [DN11, Corollary 1.2]. This completes the proof. \square

6. PROOF OF THEOREM 1.5

In this section, we show Theorem 1.5.

In [Og10], we find the following:

Theorem 6.1. *There is a (necessarily non-projective) K3 surface S with an automorphism g satisfying the following properties:*

- (1) $\text{Aut}(S) = \langle g \rangle \simeq \mathbb{Z}$.
- (2) *The topological entropy of g is the natural logarithm of the Salem number*

$$\alpha = 1.200026\dots,$$

which is the unique real roots > 1 of the following irreducible monic polynomial in the polynomial ring $\mathbb{Z}[x]$:

$$\varphi_{14}(x) = x^{14} - x^{11} - x^{10} + x^7 - x^4 - x^3 + 1.$$

- (3) *Complete irreducible curves on S are exactly 8 smooth rational curves C_i ($1 \leq i \leq 8$) and*

$$\text{NS}(S) = \mathbb{Z}\langle C_i \rangle_{i=1}^8 \simeq E_8,$$

i.e., the intersection number (C_i, C_j) ($1 \leq i < j \leq 8$) is 0 or 1 and it is 1 exactly when (i, j) is in

$$\{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 7), (3, 8)\}.$$

- (4) *The fixed locus S^g of g consists of $C := C_3$ and 7 isolated points P_i ($1 \leq i \leq 8$) on $\cup_{i=1}^8 C_i \setminus C$ and one point Q outside $\cup_{i=1}^8 C_i$.*

We show that the K3 surface S in Theorem 6.1 satisfies the conditions (1) and (2) in Theorem 1.5.

As the Néron-Severi lattice $(\text{NS}(S), (*, **))$ is negative definite by Theorem 6.1 (3), it follows that S has no non-constant global meromorphic functions. In particular, S is non-projective.

Consider the curve $C := C_3 \simeq \mathbb{P}^1$. We show that C satisfies the condition (1) in Theorem 1.5.

As the Dynkin diagram E_8 has no non-trivial automorphism, $\text{Aut}(S)$ preserves each curve C_i ($1 \leq i \leq 8$). As $C = C_3$ meets C_2, C_4, C_8 and $C_3 \cap C_4, C_3 \cap C_4, C_3 \cap C_8$ are three mutually different points on $C_3 \simeq \mathbb{P}^1$, it follows that $\text{Aut}(S) = \text{Ine}(C)$. Then $g \in \text{Ine}(C)$ and $h_{\text{top}}(g) > 0$ by Theorem 6.1 (2).

Consider the curve $D := C_8 \simeq \mathbb{P}^1$. We show that D satisfies the condition (2) in Theorem 1.5.

As $\text{Aut}(S) = \langle g \rangle$ and g preserves D , it follows that $\text{Aut}(S) = \text{Dec}(D)$.

Now consider $\text{Ine}(D)$. As g preserves each curves C_i and C_i ($1 \leq i \leq 8$) generate $\text{NS}(S)$, it follows that $g^* = \text{id}$ on $\text{NS}(S)$. On the other hand, as $\text{NS}(S)$ is negative definite of rank 8, the transcendental lattice $T(S)$ is of rank 14, which is preserved by g . Then by Theorem 6.1 (2), the characteristic polynomial of $g^*|_{T(S)}$ is exactly $\varphi_{14}(x)$. As $\varphi_{14}(x)$ is irreducible over \mathbb{Z} and g preserves $\mathbb{C}\omega_S \subset T(S)_{\mathbb{C}}$, it follows that $g^*\omega_S = \delta\omega_S$ for some Galois conjugate δ of α . Note that δ is *not* a root of unity, as δ is a Galois conjugate of $\alpha > 1$, while $|\delta| = 1$ by $(f^*\omega_S, f^*\overline{\omega_S}) = (\omega_S, \overline{\omega_S}) \neq 0$. Here we denote by $\overline{*}$ the complex conjugate of $*$ and $|\delta| := \sqrt{\delta\overline{\delta}} \in \mathbb{R}_{\geq 0}$.

Let P be the intersection point of $C = C_3$ and $D = C_8$. We denote by x an affine coordinate of $D \simeq \mathbb{P}^1$ with $x(P) = 0$. As $g|_C = \text{id}_C$, C and D meet at P transversally and $g^*\omega_S = \delta\omega_S$, it follows that $(g|_D)^*x = \delta x$, i.e., $g|_D \in \text{Aut}(D)$ is the multiplication automorphism $\delta \cdot$ by δ . As $\text{Aut}(S) = \langle g \rangle$ by Theorem 6.1 (1) and δ is not a root of unity, it follows that the natural restriction homomorphism

$$\tau : \langle g \rangle = \text{Aut}(S) = \text{Dec}(D) \rightarrow \text{Aut}(D) ; g^n \mapsto (\delta \cdot)^n = \delta^n.$$

is injective. Hence $\text{Ine}(D) = \text{Ker } \tau = \{\text{id}_S\}$. This completes the proof of Theorem 1.5.

REFERENCES

- [BHPV04] Barth, W., Hulek, K., Peters, C., Van de Ven, *Compact complex surfaces*. Second enlarged edition. Springer Verlag, Berlin-Heidelberg (2004).
- [BK09] Bedford, E., Kim, K., *Dynamics of rational surface automorphisms: linear fractional recurrences*, J. Geom. Anal. **19** (2009) 553–583.
- [Ca01] Cantat, S., *Dynamique du groupe d'automorphismes des surfaces K3*, Transform. Groups **6** (2001) 201–214.
- [DS05] Dinh, T.-C., Sibony, N., *Une borne supérieure de l'entropie topologique d'une application rationnelle*, Ann. of Math., **161** (2005) 1637–1644.
- [DS16] Dinh, T.-C., Sibony, N., *Equidistribution problems of complex dynamics in higher dimension*, arXiv:1611.03598
- [DN11] Dinh, T.-C., Nguyen V.-A., : *Comparison of dynamical degrees for semi-conjugate meromorphic maps*, Comment. Math. Helv. **86** (2011) 817–840.
- [DZ01] Dolgachev, I. V., Zhang, De-Qi, *Coble rational surfaces*, Amer. J. Math. **123** (2001) 79–114.
- [Gr87] M. Gromov, *Entropy, homology and semialgebraic geometry*, Astérisque **145–146** (1987) 225–240.

- [KH95] Katok, A., Hasselblatt, B., *Introduction to the modern theory of dynamical systems*, Cambridge University Press, 1995.
- [Ko63] Kodaira, K., *On compact analytic surfaces. II*, Ann. of Math. **77** (1963), 563–626.
- [Mc02] McMullen, C. T., *Dynamics on K3 surfaces: Salem numbers and Siegel disks*, J. Reine Angew. Math. **545** (2002), 201–233.
- [Mc07] McMullen, C. T., *Dynamics on blowups of the projective plane*, Publ. Math. Inst. Hautes Études Sci. **105** (2007) 49–89.
- [Ni81] Nikulin, V., *Quotient-groups of groups of automorphisms of hyperbolic forms by subgroups generated by 2-reflections, Algebro-geometric applications*, Current Problems in Mathematics, **18** (1981) 3–114.
- [Og89] Oguiso, K., *On Jacobian fibrations on the Kummer surfaces of the product of nonisogenous elliptic curves*, J. Math. Soc. Japan **41** (1989) 651–680.
- [Og07] Oguiso, K., *Automorphisms of hyperkähler manifolds in the view of topological entropy*, 173–185, Contemp. Math., **422** 2007.
- [Og08] Oguiso, K., *Mordell-Weil groups of a hyperkähler manifold - a question of F. Campana*, Publ. Res. Inst. Math. Sci. **44** (2008) 495–506.
- [Og10] Oguiso, K., *The third smallest Salem number in automorphisms of K3 surfaces*, 331–360, Adv. Stud. Pure Math., **60**, 2010.
- [OZ96] Oguiso, K., Zhang, De-Qi, *On the most algebraic K3 surfaces and the most extremal log Enriques surfaces*, Amer. J. Math. **118** (1996) 1277–1297.
- [Tr15] Truong, T.T., : *(Relative) dynamical degrees of rational maps over an algebraic closed field*, arXiv:1501.01523.
- [Ue75] Ueno, K., *Classification theory of algebraic varieties and compact complex spaces*, Lecture Notes in Mathematics, **439**, Springer-Verlag, 1975.
- [Vi83] Vinberg, É. B., *The two most algebraic K3 surfaces*, Math. Ann. **265** (1983) 1–21.
- [Zh09] Zhang, D.-Q., *A theorem of Tits type for compact Kähler manifolds*, Invent. Math. **176** (2009) 449–459.

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